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Embedding theorems and integration operators on Bergman spaces with rapidly decreasing weights

Jordi Pau [a,1], José Ángel Peláez [b,*,1]

[a] Departament de Matemàtica Aplicada i Analisi, Universitat de Barcelona, 08007 Barcelona, Spain

[b] Departamento de Análisis Matemático, Universidad de Málaga, Campus de Teatinos, 29071 Málaga, Spain

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Abstract

We completely describe those positive Borel measures μ in the unit disc $\mathbb{D}$ such that the Bergman space $A^p(w) \subset L^q(\mu)$, $0 < p, q < \infty$, where w belongs to a large class $\mathcal{W}$ of rapidly decreasing weights which includes the exponential weights $w_\alpha(r) = \exp(\frac{-1}{(1-r)^\alpha})$, $\alpha > 0$, and some double exponential type weights.

As an application of that result, we characterize the boundedness and the compactness of $T_g : A^p(w) \to A^q(w)$, $0 < p, q < \infty$, $w \in \mathcal{W}$, where T_g is the integration operator

$$(T_g f)(z) = \int_0^z f(\zeta) g'(\zeta)\, d\zeta.$$

The particular choice of the weight $w_\alpha(r)$ answers an open question posed by A. Aleman and A. Siskakis. We also describe those analytic functions in $\mathbb{D}$ for which T_g belongs to the Schatten p-class of $A^2(w)$, $0 < p < \infty$.

Keywords: Integration operators; Bergman spaces; Carleson measures; Schatten classes

* Corresponding author.

E-mail addresses: jordi.pau@ub.edu (J. Pau), japelaez@uma.es (J.Á. Peláez).

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1. Introduction and main results

Let $\mathbb{D}$ be the unit disc in the complex plane, $dm(z) = \frac{dx\,dy}{\pi}$ be the normalized area measure on $\mathbb{D}$, and denote by $H(\mathbb{D})$ the space of all analytic functions in $\mathbb{D}$. A positive function $w(r)$, $0 \leqslant r < 1$, which is integrable in $(0, 1)$, will be called a *weight function*. We extend w to $\mathbb{D}$ setting $w(z) = w(|z|)$, $z \in \mathbb{D}$. For $0 < p < \infty$, the weighted Bergman space $A^p(w)$ is the space of functions $f \in H(\mathbb{D})$ such that

$$\|f\|^p_{A^p(w)} = \int_{\mathbb{D}} |f(z)|^p w(z)\, dm(z) < \infty.$$

Note that $A^p(w)$ is the closed subspace of $L^p(\mathbb{D}, w\,dm)$ consisting of analytic functions. If $w(r) = (1-r)^\alpha$, $\alpha > -1$, the standard Bergman spaces A^p_α are obtained.

In this work we are going to study Carleson measures and integration operators on Bergman spaces with rapidly decreasing weights, that is, weights that are going to decrease faster than any standard weight $(1-r)^\alpha$, $\alpha > 0$. Concretely, we consider a decreasing weight of the form $w(z) = e^{-\varphi(z)}$, where $\varphi \in C^2(\mathbb{D})$ is a radial function such that $\Delta\varphi(z) \geqslant B_\varphi > 0$ for some positive constant B_φ depending only on the function φ. Here Δ denotes the standard Laplace operator.

We will assume that $(\Delta\varphi(z))^{-1/2} \asymp \tau(z)$, where $\tau(z)$ is a radial positive function that decreases to 0 as $|z| \to 1^-$, and $\lim_{r\to 1^-} \tau'(r) = 0$.

Furthermore, we shall also suppose that either there exists a constant $C > 0$ such that $\tau(r)(1-r)^{-C}$ increases for r close to 1 or

$$\lim_{r\to 1^-} \tau'(r) \log\frac{1}{\tau(r)} = 0.$$

If the weight w satisfies all the previous conditions, we shall say that the weight w belongs to the class $\mathcal{W}$. The class $\mathcal{W}$ includes (see Section 7) the exponential type weights

$$w_{\gamma,\alpha}(r) = (1-r)^\gamma \exp\left(\frac{-c}{(1-r)^\alpha}\right), \quad \gamma \geqslant 0,\ \alpha > 0,\ c > 0,$$

and the double exponential weights

$$w(r) = \exp\left(-\gamma \exp\left(\frac{\beta}{(1-r)^\alpha}\right)\right), \quad \alpha, \beta, \gamma > 0.$$

For the weights w considered in this paper, the point evaluations L_a at the point $a \in \mathbb{D}$, are bounded linear functionals on $A^2(w)$. Therefore, there are reproducing kernels $K_a \in A^2(w)$ with $\|L_a\| = \|K_a\|_{A^2(w)}$ such that

$$L_a f = \langle f, K_a\rangle = \int_{\mathbb{D}} f(z)\overline{K_a(z)} w(z)\, dm(z), \quad f \in A^2(w).$$

Some basic properties of the Bergman spaces $A^p(w)$, $w \in \mathcal{W}$, are not yet well understood and have attracted some attention in recent years. The interest in such spaces arises from the fact that the usual techniques for the standard Bergman spaces fail to work in this context. For example, the natural Bergman projection

$$Pf(a) = \int_{\mathbb{D}} f(z)\overline{K_a(z)}w(z)\,dm(z), \quad a \in \mathbb{D},$$

is not necessarily bounded on $L^p(\mathbb{D}, w\,dm)$ if $p \neq 2$ (see [9]). This carries that the dual spaces of $A^p(w)$, $w \in \mathcal{W}$ have not been identified.

Let X be a space of analytic functions on the unit disc $\mathbb{D}$. A positive Borel measure μ in $\mathbb{D}$ is said to be a q-Carleson measure for X if the embedding $X \subset L^q(\mu)$, $0 < q < \infty$, is continuous. After the pioneering works of L. Carleson (see [7] and [8]), there has been a great amount of research on this topic, and these measures have found many applications in other related areas. A description of those measures have been obtained for several spaces of analytic functions (see e.g. [5,11,14,19]). Here we obtain a complete description of the q-Carleson measures for $A^p(w)$, $0 < p, q < \infty$, for weights w in the class $\mathcal{W}$.

Let $D(a, r)$ be the Euclidean disc centered at a with radius $r > 0$, and for easy of notation, for any $\delta > 0$ we write $D(\delta\tau(a))$ for the disc $D(a, \delta\tau(a))$.

Theorem 1. *Let $w \in \mathcal{W}$ and let μ be a finite positive Borel measure on $\mathbb{D}$.*

(I) *Let $0 < p \leqslant q < \infty$.*
  (a) *The embedding $I_d : A^p(w) \to L^q(\mu)$ is bounded if and only if for each sufficiently small $\delta > 0$ we have*

$$K_{\mu,w} := \sup_{a\in\mathbb{D}} \frac{1}{\tau(a)^{2q/p}} \int_{D(\delta\tau(a))} w(z)^{-q/p}\,d\mu(z) < \infty. \tag{1}$$

  Moreover, if any of the two equivalent conditions holds, then

$$K_{\mu,w} \asymp \|I_d\|^q_{A^p(w)\to L^q(\mu)}.$$

  (b) *The embedding $I_d : A^p(w) \to L^q(\mu)$ is compact if and only if for each sufficiently small $\delta > 0$ we have*

$$\lim_{r\to 1^-} \sup_{|a|>r} \frac{1}{\tau(a)^{2q/p}} \int_{D(\delta\tau(a))} w(z)^{-q/p}\,d\mu(z) = 0. \tag{2}$$

(II) *Let $0 < q < p < \infty$. The following conditions are equivalent*:
  (a) *$I_d : A^p(w) \to L^q(\mu)$ is compact*;
  (b) *$I_d : A^p(w) \to L^q(\mu)$ is bounded*;
  (c) *For each sufficiently small $\delta > 0$, the function*

$$z \mapsto \frac{1}{\tau(z)^2} \int_{D(\delta\tau(z))} w(\zeta)^{-q/p}\,d\mu(\zeta)$$

  belongs to $L^{\frac{p}{p-q}}(\mathbb{D}, dm)$.

For the standard Bergman spaces A^p_α, the statement of Theorem 1 also holds, and in that case the condition can be simplified (see [17, Theorem 2.2] for the case $0 < p \leqslant q < \infty$, and [20] for the case $0 < q < p < \infty$). Related results can be found in [21] and [22].

The above result can be used in order to study several related questions. Here we put our attention on the operators T_g defined by

$$(T_g f)(z) = \int_0^z f(\zeta) g'(\zeta)\, d\zeta,$$

where g is an analytic function on $\mathbb{D}$. The boundedness and compactness of T_g on classical spaces has attracted a lot of attention in recent years (see [2,3] for Hardy spaces, [4,10,25] for weighted Bergman spaces, and [13,14] for Dirichlet-type spaces). We also mention the surveys [1] and [27] for an account of results and open questions on the operator T_g. Note that as special choices of the symbol g one gets several important operators: the Volterra operator (T_g with $g(z) = z$) and the Cesáro operator (T_g with $g(z) = \log(1/(1-z))$). In particular, A. Aleman and A. Siskakis proved in [4] the following result.

Theorem A. *Suppose that w is a weight satisfying the following conditions*:

$(P1)$ *There is a positive constant C such that*

$$\frac{1}{w(r)} \int_r^1 w(s)\, ds \leqslant C(1-r).$$

$(P2)$ *There are $s \in (0,1)$ and a positive constant C such that*

$$w(sr + 1 - s) \geqslant C w(r), \quad 0 < r < 1.$$

Then for $1 \leqslant p < \infty$ T_g is bounded (*compact*) *on $A^p(w)$ if and only if g belongs to the Bloch space* (*little Bloch space*).

Recall that an analytic function f in $\mathbb{D}$ belongs to the Bloch space if $\sup_{z\in\mathbb{D}}(1 - |z|)|f'(z)| < \infty$, and f belongs to the little Bloch space if $\lim_{|z|\to 1^-}(1-|z|)|f'(z)| = 0$.

The large class of radial weights w satisfying conditions $(P1)$ and $(P2)$ of Theorem A includes the standard weights $w(r) = (1-r)^\alpha$, $\alpha > -1$, but excludes the exponential ones

$$w_\alpha(r) = \exp\left(\frac{-c}{(1-r)^\alpha}\right), \quad c > 0,\ \alpha > 0 \tag{3}$$

(they do no satisfy condition $(P2)$). In relation with the exponential weights, A. Aleman and A. Siskakis in [4] proved that, for $1 \leqslant p < \infty$, T_g is bounded on $A^p(w_\alpha)$ if

$$\big(1-|z|\big)^{\alpha+1}\big|g'(z)\big| = O(1), \quad \text{as } |z| \to 1,$$

and the operator T_g is compact on $A^p(w_\alpha)$ whenever the corresponding "little oh" condition holds, and raised the open problem of whether this condition is necessary for the boundedness (compactness) of T_g. A positive answer for $p = 2$, $c > 0$ and $\alpha \in (0, 1]$ has been given recently by Dostanić in [10] by using precise and specific techniques which involve the exponential weight defined in (3). In this paper we shall give a positive answer to that question (see Theorem 2 below). Indeed, we completely describe the boundedness and compactness of $T_g : A^p(w) \to A^q(w)$, $0 < p, q < \infty$, for weights $w \in \mathcal{W}$.

In the proof of Theorem A given in [4], two facts play an essential role. First, for some $0 < s < 1$ the composition operator induced by the symbol $\psi_s(z) = sz - s + 1$ is bounded on $A^p(w)$. Since this does not remain true for rapidly decreasing weights such that w_α (see [15, Theorem 1.1]) their method cannot be applied in our case. The second key step in the proof of Theorem A consists of proving that

$$f \in A^p(w) \quad \Leftrightarrow \quad \big(1 - |z|\big)\big|f'(z)\big| \in L^p(w), \tag{4}$$

consequently it will be useful for our class of weights to establish a result analogous to (4) replacing the quantity $(1 - |z|)$ for a suitable distortion function. So, following Siskakis [26], for a given weight w, we define the *distortion function* of w by

$$\psi_w(r) = \frac{1}{w(r)} \int_r^1 w(u)\,du, \quad 0 \leqslant r < 1.$$

The next result is the case $q = p$ of Theorem 1.1 of [24].

Theorem B. *Suppose that w is a differentiable weight, and there is $L > 0$ such that*

$$\sup_{0<r<1} \frac{w'(r)}{w(r)^2} \int_r^1 w(x)\,dx \leqslant L, \tag{5}$$

then for each $p \in (0, \infty)$ and $g \in H(\mathbb{D})$

$$\int_{\mathbb{D}} \big|g(z)\big|^p w(z)\,dm(z) \asymp \big|g(0)\big|^p + \int_{\mathbb{D}} \big|g'(z)\big|^p \psi_w(z)^p w(z)\,dm(z).$$

It is clear that condition (5) is satisfied for any decreasing differentiable weight.

Finally, we are going to state the promised result about the description of the boundedness and compactness of $T_g : A^p(w) \to A^q(w)$, $0 < p, q < \infty$, $w \in \mathcal{W}$.

Theorem 2. *Let $0 < p, q < \infty$, $g \in H(\mathbb{D})$, and $w \in \mathcal{W}$ with*

$$\Delta\varphi(z) \asymp \big(\big(1 - |z|\big)^t \psi_w(z)\big)^{-1}, \quad z \in \mathbb{D}, \text{ for some } t \geqslant 1. \tag{6}$$

(I) (a) *T_g is bounded on $A^p(w)$ if and only if*

$$\sup_{z\in\mathbb{D}} \psi_w(z)\big|g'(z)\big| < \infty.$$

(b) *T_g is compact on $A^p(w)$ if and only if*

$$\lim_{r\to 1^-}\sup_{|z|>r}\psi_w(z)\big|g'(z)\big|=0.$$

(II) *Let $0<p<q<\infty$. Then $T_g : A^p(w)\to A^q(w)$ is bounded if and only if g is constant.*
(III) *Let $0<q<p<\infty$. The following conditions are equivalent*:
  (a) *$T_g : A^p(w)\to A^q(w)$ is compact*;
  (b) *$T_g : A^p(w)\to A^q(w)$ is bounded*;
  (c) *The function $g\in A^r(w)$, where $r=\frac{pq}{p-q}$.*

The corresponding result for the standard Bergman spaces A^p_α can be found in [4] and [25]. The particular choice of the exponential weight w_α defined in (3) solves the problem posed in [4, p. 353], since the distorsion function of w_α is comparable to $(1-|z|)^{1+\alpha}$ (see e.g. [26, Example 3.2]).

If T is a compact operator on a separable Hilbert space H, then there exist orthonormal sets $\{e_n\}$ and $\{\sigma_n\}$ in H such that

$$Tx=\sum_n \lambda_n\langle x,e_n\rangle\sigma_n,\quad x\in H,$$

where λ_n is the nth singular value of T.

Given $0<p<\infty$, let $\mathcal{S}_p(H)$ denote the Schatten p-class of operators on H. The class $\mathcal{S}_p(H)$ consists of those compact operators T on H with its sequence of singular numbers λ_n belonging to ℓ^p, the p-summable sequence space. It is also well known that, if λ_n are the singular numbers of an operator T, then

$$\lambda_n=\lambda_n(T)=\inf\big\{\|T-K\|:\ \operatorname{rank}K\leqslant n\big\}.$$

Thus finite rank operators belong to every $\mathcal{S}_p(H)$, and the membership of an operator in $\mathcal{S}_p(H)$ measures in some sense the size of the operator. It is also clear that the use of two equivalent norms in H does not change the class $\mathcal{S}_p(H)$. In the case when $1\leqslant p<\infty$, $\mathcal{S}_p(H)$ is a Banach space with the norm

$$\|T\|_p=\left(\sum_n|\lambda_n|^p\right)^{1/p},$$

while for $0<p<1$ we have the following inequality

$$\|T+S\|_p^p\leqslant\|T\|_p^p+\|S\|_p^p.$$

We refer to [28] for more information about $\mathcal{S}_p(H)$.

In [3], using a result D. Luecking [18] concerning the Schatten classes of certain Toeplitz operators (see also [2] and [16] for related results), a description of those $g\in H(\mathbb{D})$ for which $T_g\in\mathcal{S}_p(A^2_\alpha)$ is obtained. In this paper we give a direct proof of the following result.

Theorem 3. *Let $g \in H(\mathbb{D})$ and $w \in \mathcal{W}$ satisfying* (6)*.*

(a) *If $1 < p < \infty$ then $T_g \in \mathcal{S}_p(A^2(w))$ if and only if*

$$\psi_w |g'| \in L^p(\mathbb{D}, \Delta\varphi \, dm).$$

(b) *If $0 < p \leqslant 1$ then $T_g \in \mathcal{S}_p(A^2(w))$ if and only if g is constant.*

The paper is organized as follows: Section 2 is devoted to some preliminaries needed for the proofs of the main results. In Section 3 we construct the test functions which are used in the proof of Theorem 1 in Section 4. We prove Theorem 2 in Section 5 and Theorem 3 in Section 6. Finally, in Section 7 several examples of weights w in the class $\mathcal{W}$ are given.

Throughout the paper, the letter C will denote an absolute constant whose value may change at different occurrences. We also use the notation $a \lesssim b$ to indicate that there is a constant $C > 0$ with $a \leqslant Cb$, and the notation $a \asymp b$ means that $a \lesssim b$ and $b \lesssim a$.

2. Preliminaries

In this section we present some previous results, that can be of independent interest, which are needed to prove the main results.

Let τ be a positive function on $\mathbb{D}$. We say that $\tau \in \mathcal{L}$ if there exist constants $c_0 > 0$ and $c_1 > 0$ such that

$$\tau(z) \leqslant c_0\big(1 - |z|\big) \quad \text{for } z \in \mathbb{D}; \tag{7}$$

$$\big|\tau(z) - \tau(\zeta)\big| \leqslant c_1 |z - \zeta| \quad \text{for } z, \zeta \in \mathbb{D}. \tag{8}$$

Throughout this paper, we will always use the notation

$$m_\tau = \frac{\min(1, c_0^{-1}, c_1^{-1})}{4},$$

where c_0 and c_1 are the constants in (7) and (8).

Lemma 2.1. *Suppose that $\tau \in \mathcal{L}$, $0 < \delta \leqslant m_\tau$ and $a \in \mathbb{D}$. Then,*

$$\frac{1}{2}\tau(a) \leqslant \tau(z) \leqslant 2\tau(a) \quad \text{if } z \in D\big(\delta\tau(a)\big).$$

Proof. Note that, by condition (8) we have

$$\tau(a) \leqslant \tau(z) + c_1|z - a| \leqslant \tau(z) + \frac{1}{4}\tau(a) \quad \text{if } |z - a| \leqslant \delta\tau(a).$$

Therefore $\tau(a) \leqslant 2\tau(z)$ if $|z - a| \leqslant \delta\tau(a)$. Similarly it can be proved that $\tau(z) \leqslant 2\tau(a)$. □

The following result, where the fact that $|f(z)|^p w(z)$ verifies a certain sub-mean-value property is proved, will play an essential role in the proof of the main theorems of this paper.

Lemma 2.2. *Let φ be a subharmonic function, $w = e^{-\varphi}$, and let $\tau \in \mathcal{L}$ such that $\tau(z)^2 \Delta\varphi(z) \leqslant c_2$ for some constant $c_2 > 0$. If $\beta \in \mathbb{R}$, there exists a constant $M \geqslant 1$ such that*

$$\left|f(a)\right|^p w(a)^\beta \leqslant \frac{M}{\delta^2 \tau(a)^2} \int\limits_{D(\delta\tau(a))} \left|f(z)\right|^p w(z)^\beta \, dm(z),$$

for all $0 < \delta \leqslant m_\tau$ and $f \in H(\mathbb{D})$.

Proof. By Green's formula we have

$$\frac{1}{2\pi} \int\limits_{|\zeta - a| = r} \varphi(\zeta)\, d\zeta = r\varphi(a) + \frac{r}{2} \int\limits_{|z-a| \leqslant r} \Delta\varphi(z) \log\frac{r}{|z-a|}\, dm(z).$$

Integrating this identity between zero and $\delta\tau(a)$, using that $\tau(z)^2 \Delta\varphi(z) \leqslant c_2$, and Lemma 2.1, we obtain

$$\begin{aligned}
\frac{1}{\tau(a)^2} \int\limits_{D(\delta\tau(a))} \varphi(\zeta)\, dm(\zeta) &\leqslant \delta^2 \varphi(a) + \frac{c_2}{\tau(a)^2} \int\limits_0^{\delta\tau(a)} \int\limits_{|z-a| \leqslant r} \tau(z)^{-2} \log\frac{r}{|z-a|}\, dm(z)\, r\, dr \\
&\leqslant \delta^2 \varphi(a) + \frac{4c_2}{\tau(a)^4} \int\limits_0^{\delta\tau(a)} \int\limits_{|z-a| \leqslant r} \log\frac{r}{|z-a|}\, dm(z)\, r\, dr \\
&= \delta^2 \varphi(a) + \frac{2c_2}{\tau(a)^4} \int\limits_0^{\delta\tau(a)} r^3\, dr \\
&= \delta^2 \varphi(a) + \frac{c_2 \delta^4}{2}.
\end{aligned}$$

So if $\beta > 0$, rewriting the previous equation in terms of the weight w, and multiplying by β, we obtain

$$\log w(a)^\beta \leqslant \frac{1}{\delta^2 \tau(a)^2} \int\limits_{D(\delta\tau(a))} \log w(\zeta)^\beta\, dm(\zeta) + \log M, \tag{9}$$

with $M = \exp(\beta c_2 \delta^2 / 2) \geqslant 1$. Also, the subharmonicity of $\log|f|$ gives

$$\log\left|f(a)\right|^p \leqslant \frac{1}{\delta^2 \tau(a)^2} \int\limits_{D(\delta\tau(a))} \log\left|f(z)\right|^p\, dm(z). \tag{10}$$

Now, adding Eqs. (10) and (9), and using the arithmetic–geometric inequality we get the desired result.

If $\beta \leqslant 0$, then w^β is a logarithmic subharmonic function, so (9) holds with $M = 1$, and arguing as in the previous case the conclusion is obtained. This finishes the proof. □

We note that if a weight w belongs to the class $\mathcal{W}$, then its associated function $\tau(z)$ belongs to the class $\mathcal{L}$. Thus Lemma 2.2 proves that for weights w in the class $\mathcal{W}$, the point evaluations L_a are bounded linear functionals on $A^p(w)$. Another consequence is that norm convergence implies uniform convergence on compact subsets of $\mathbb{D}$. It follows that for $w \in \mathcal{W}$, the space $A^p(w)$ is complete.

The next result proves that the weights in the class $\mathcal{W}$ decrease faster than the standard weights $(1-|z|)^\alpha$, $\alpha > 0$.

Lemma 2.3. *Consider a weight of the form $w(z) = e^{-\varphi(z)}$, where $\varphi \in C^2(\mathbb{D})$ is a radial function with $(\Delta\varphi(z))^{-1/2} \asymp \tau(z)$, and $\tau(z)$ is a radial positive differentiable function that decreases to* 0 *as $|z| \to 1^-$. If $\lim_{r\to 1} \tau'(r) = 0$, then*

$$\lim_{|z|\to 1^-} \frac{w(z)}{\tau(z)^\alpha} = 0, \quad \textit{for each } \alpha > 0.$$

Proof. We may assume without loss of generality that $\varphi(0) = \varphi'(0) = 0$. Since

$$\frac{w(z)}{\tau(z)^\alpha} = e^{-\varphi(z)-\alpha\log\tau(z)},$$

it is enough to show that

$$\lim_{|z|\to 1^-} \frac{\varphi(z)}{\log\frac{1}{\tau(z)}} = +\infty.$$

Write $r = |z|$. Use the fact that $\tau'(r)$ is negative in some interval $(r_0, 1)$, the formula

$$\varphi'(r) = \frac{1}{r}\int_0^r s\,\Delta\varphi(s)\,ds,$$

and the assumption $\lim_{r\to 1} \tau(r) = \lim_{r\to 1} \tau'(r) = 0$, to obtain

$$\varphi'(r) \geqslant C\int_{r_0}^r \frac{s\,ds}{\tau(s)^2} \geqslant C\int_{r_0}^r \frac{(-\tau'(s))\,ds}{\tau(s)^2} \geqslant \frac{C}{\tau(r)}$$

for r close to 1. This, together with Bernouilli–l'Hôpital theorem gives

$$\lim_{r\to 1^-} \frac{\varphi(r)}{\log\frac{1}{\tau(r)}} = \lim_{r\to 1^-} \frac{\varphi'(r)\tau(r)}{-\tau'(r)} \geqslant \lim_{r\to 1^-} \frac{C}{-\tau'(r)} = +\infty.$$

This finishes the proof. $\square$

The following lemma on coverings is due to Oleinik, see [21].

Lemma A. *Let τ be a positive function in $\mathbb{D}$ in the class $\mathcal{L}$, and let $\delta \in (0, m_\tau)$. Then there exists a sequence of points $\{z_j\} \subset \mathbb{D}$, such that the following conditions are satisfied:*

(i) $z_j \notin D(\delta\tau(z_k))$, $j \neq k$.
(ii) $\bigcup_j D(\delta\tau(z_j)) = \mathbb{D}$.
(iii) $\tilde{D}(\delta\tau(z_j)) \subset D(3\delta\tau(z_j))$, *where* $\tilde{D}(\delta\tau(z_j)) = \bigcup_{z\in D(\delta\tau(z_j))} D(\delta\tau(z))$, $j = 1, 2, \ldots$.
(iv) $\{D(3\delta\tau(z_j))\}$ *is a covering of* $\mathbb{D}$ *of finite multiplicity* N.

3. Test functions

It is known that having an appropriate family of test functions in a space of analytic functions X can be a good help in order to characterize the q-Carleson measures for X. In this section we will do the job for the spaces $A^p(w)$. The following result, partially proved in [6], will be a key in the proof of Theorem 1.

Lemma 3.1. *Assume that* $0 < p < \infty$, $n \in \mathbb{N} \setminus \{0\}$ *with* $np \geqslant 1$ *and* $w \in \mathcal{W}$. *Then, there is a number* $\rho_0 \in (0, 1)$ *such that for each* $a \in \mathbb{D}$ *with* $|a| \geqslant \rho_0$, *there is a function* $F_{a,n,p}$ *analytic in* $\mathbb{D}$ *with*

$$\left|F_{a,n,p}(z)\right|^p w(z) \asymp 1 \quad \text{if } |z-a| < \tau(a), \tag{11}$$

and

$$\left|F_{a,n,p}(z)\right| w(z)^{1/p} \lesssim \min\left(1, \frac{\min(\tau(a), \tau(z))}{|z-a|}\right)^{3n}, \quad z \in \mathbb{D}. \tag{12}$$

Proof. If $1 \leqslant p < \infty$, and $n = 1$ the result follows directly from Lemma 3.3 in [6]. Now, if $0 < p < \infty$ and $np \geqslant 1$ applying the previous case, we have

$$\left|F_{a,1,np}(z)\right|^{np} w(z) \asymp 1 \quad \text{if } |z-a| < \tau(a),$$

and

$$\left|F_{a,1,np}(z)\right| w(z)^{\frac{1}{np}} \lesssim \min\left(1, \frac{\min(\tau(a), \tau(z))}{|z-a|}\right)^{3}, \quad z \in \mathbb{D}.$$

That is, if we choose $F_{a,n,p} = F^n_{a,1,np}$

$$\left|F_{a,n,p}(z)\right|^p w(z) \asymp 1 \quad \text{if } |z-a| < \tau(a),$$

and

$$\left|F_{a,n,p}(z)\right| w(z)^{1/p} \lesssim \min\left(1, \frac{\min(\tau(a), \tau(z))}{|z-a|}\right)^{3n}, \quad z \in \mathbb{D}.$$

This finishes the proof. □

As an immediate consequence of that lemma, as it is noticed also in [6], we get an estimate for the reproducing kernels of $A^2(w)$.

Corollary 1. *Let $w \in \mathcal{W}$. There is a number $\rho_0 \in (0,1)$ such that*

(i) *for $0 < p < \infty$ and $n \in \mathbb{N} \setminus \{0\}$ with $np \geqslant 1$ the function $F_{a,n,p}$ in Lemma 3.1 belongs to $A^p(w)$ with*

$$\|F_{a,n,p}\|^p_{A^p(w)} \asymp \tau(a)^2, \quad \rho_0 \leqslant |a| < 1;$$

(ii) *the reproducing kernel K_a of $A^2(w)$ satisfies the estimate*

$$\|K_a\|^2_{A^2(w)} w(a) \asymp \tau(a)^{-2}, \quad \rho_0 \leqslant |a| < 1.$$

Proof. Let $a \in \mathbb{D}$ with $\rho_0 \leqslant |a| < 1$, and consider the functions $F_{a,n,p}$ obtained in Lemma 3.1. Write

$$R_k(a) = \bigl\{z \in \mathbb{D}\colon\ 2^{k-1}\tau(a) < |z-a| \leqslant 2^k \tau(a)\bigr\}, \quad k = 1, 2 \ldots.$$

Note that (11) gives

$$\int\limits_{|z-a|<\tau(a)} \bigl|F_{a,n,p}(z)\bigr|^p w(z)\, dm(z) \asymp \tau(a)^2,$$

and, by (12) and the fact that $3np > 2$,

$$\begin{aligned}
\int\limits_{|z-a|>\tau(a)} \bigl|F_{a,n,p}(z)\bigr|^p w(z)\, dm(z) &\leqslant \sum_{k=1}^{\infty} \int\limits_{R_k(a)} \bigl|F_{a,n,p}(z)\bigr|^p w(z)\, dm(z) \\
&\lesssim \tau(a)^{3np} \sum_{k=1}^{\infty} \int\limits_{R_k(a)} \frac{dm(z)}{|z-a|^{3np}} \\
&\lesssim \sum_{k=1}^{\infty} 2^{-3npk} m\bigl(R_k(a)\bigr) \\
&\lesssim \tau(a)^2.
\end{aligned}$$

Therefore $F_{a,n,p} \in A^p(w)$ with $\|F_{a,n,p}\|^p_{A^p(w)} \asymp \tau(a)^2$, which gives (i).

The use of Lemma 2.2 (with $\beta = 1$) gives the upper estimate of (ii),

$$\|K_a\|^2_{A^2(w)} w(a) \lesssim \tau(a)^{-2}.$$

On the other hand, the functions $F_{a,1,2}$ obtained from the previous lemma satisfy (by (i)) that $F_{a,1,2} \in A^2(w)$ with $\|F_{a,1,2}\|^2_{A^2(w)} \asymp \tau(a)^2$, and by (11) this gives

$$\bigl|F_{a,1,2}(a)\bigr|^2 \asymp w(a)^{-1} \asymp \bigl(w(a)\tau(a)^2\bigr)^{-1} \|F_{a,1,2}\|^2_{A^2(w)}.$$

Since $\|K_a\|_{A^2(w)} = \|L_a\|$, where L_a is the point evaluation functional at the point a, this proves the lower estimate of (ii). $\square$

Proposition 2. *Let $w \in \mathcal{W}$, $0 < p < \infty$ and $n \in \mathbb{N}$ such that $n \geqslant \max\{1/p, p\}$. If ρ_0 is the number given in Lemma 3.1 and $\{z_k\} \subset \mathbb{D}$ is the sequence from Lemma A, the function*

$$F(z) = \sum_{z_k:\ |z_k| \geqslant \rho_0} a_k \frac{F_{z_k,n,p}(z)}{\tau(z_k)^{2/p}}$$

belongs to $A^p(w)$ for every sequence $\{a_k\} \in \ell^p$. Moreover,

$$\|F\|_{A^p(w)} \lesssim \left(\sum_k |a_k|^p\right)^{1/p}.$$

Proof. In what follows, we shall write

$$F(z) = \sum_{z_k:\ |z_k| \geqslant \rho_0} a_k \frac{F_{z_k,n,p}(z)}{\tau(z_k)^{2/p}} = \sum_k a_k \frac{F_{z_k,n,p}(z)}{\tau(z_k)^{2/p}}.$$

If $0 < p \leqslant 1$, then bearing in mind Corollary 1, we have that

$$\begin{aligned}
\|F\|^p_{A^p(w)} &= \int_{\mathbb{D}} \left|\sum_k a_k \frac{F_{z_k,n,p}(z)}{\tau(z_k)^{2/p}}\right|^p w(z)\,dm(z) \\
&\leqslant \sum_k \frac{|a_k|^p}{\tau(z_k)^2} \|F_{z_k,n,p}\|^p_{A^p(w)} \\
&\leqslant C \sum_k |a_k|^p.
\end{aligned}$$

If $p > 1$, an application of Hölder's inequality yields

$$\left|F(z)\right|^p \leqslant \sum_k \frac{|a_k|^p}{\tau(z_k)^{2p}} \left|F_{z_k,n,p}(z)\right|^{\frac{p(n-p+1)}{n}} \left(\sum_k \tau(z_k)^2 \left|F_{z_k,n,p}(z)\right|^{p/n}\right)^{p-1}. \tag{13}$$

Now, we claim that

$$\sum_k \tau(z_k)^2 \left|F_{z_k,n,p}(z)\right|^{p/n} \lesssim \frac{\tau(z)^2}{w(z)^{1/n}}. \tag{14}$$

In order to prove (14), note first that using the estimate (11), Lemma 2.1 and (iv) of Lemma A, we deduce that

$$\sum_{\{z_k \in D(z,\delta_0\tau(z))\}} \tau(z_k)^2 \left|F_{z_k,n,p}(z)\right|^{p/n} \lesssim \frac{1}{w(z)^{1/n}} \sum_{\{z_k \in D(z,\delta_0\tau(z))\}} \tau(z_k)^2 \lesssim \frac{\tau(z)^2}{w(z)^{1/n}}. \tag{15}$$

On the other hand, an application of (12) gives

$$\sum_{\{z_k\notin D(z,\delta_0\tau(z))\}} \tau(z_k)^2 \big|F_{z_k,n,p}(z)\big|^{p/n} \lesssim \frac{\tau(z)^{3p}}{w(z)^{1/n}} \sum_{\{z_k\notin D(z,\delta_0\tau(z))\}} \frac{\tau(z_k)^2}{|z-z_k|^{3p}} = \frac{\tau(z)^{3p}}{w(z)^{1/n}} \sum_{j=0}^{\infty} \sum_{z_k\in R_j(z)} \frac{\tau(z_k)^2}{|z-z_k|^{3p}},$$

where

$$R_j(z) = \big\{\zeta \in \mathbb{D}\colon\ 2^j\delta_0\tau(z) < |\zeta - z| \leqslant 2^{j+1}\delta_0\tau(z)\big\}, \quad j = 0, 1, 2 \ldots.$$

Now observe that, using (8), it is easy to see that, for $j = 0, 1, 2, \ldots,$

$$D\big(z_k, \delta_0\tau(z_k)\big) \subset D\big(z, 5\delta_0 2^j\tau(z)\big) \quad \text{if } z_k \in D\big(z, 2^{j+1}\delta_0\tau(z)\big).$$

This fact together with the finite multiplicity of the covering (see Lemma A) gives

$$\sum_{z_k\in R_j(z)} \tau(z_k)^2 \lesssim m\big(D\big(z, 5\delta_0 2^j\tau(z)\big)\big) \lesssim 2^{2j}\tau(z)^2.$$

Therefore

$$\begin{aligned}\sum_{\{z_k\notin D(z,\delta_0\tau(z))\}} \tau(z_k)^2 \big|F_{z_k,n,p}(z)\big|^{p/n} &\lesssim \frac{\tau(z)^{3p}}{w(z)^{1/n}} \sum_{j=0}^{\infty} \sum_{z_k\in R_j(z)} \frac{\tau(z_k)^2}{|z-z_k|^{3p}} \\ &\lesssim \frac{1}{w(z)^{1/n}} \sum_{j=0}^{\infty} 2^{-3pj} \sum_{z_k\in R_j(z)} \tau(z_k)^2 \\ &\lesssim \frac{\tau(z)^2}{w(z)^{1/n}} \sum_{j=0}^{\infty} 2^{(2-3p)j} \\ &\lesssim \frac{\tau(z)^2}{w(z)^{1/n}},\end{aligned}$$

which together with (15), proves (14).

Now, joining (13) and (14), we obtain

$$\|F\|^p_{A^p(w)} \leqslant \sum_k \frac{|a_k|^p}{\tau(z_k)^{2p}} \int_{\mathbb{D}} \big|F_{z_k,n,p}(z)\big|^{\frac{p(n-p+1)}{n}} \tau(z)^{2p-2} w(z)^{\frac{n-p+1}{n}}\, dm(z).$$

So, it is enough to show that

$$\int_{\mathbb{D}} \big|F_{z_k,n,p}(z)\big|^{\frac{p(n-p+1)}{n}} \tau(z)^{2p-2} w(z)^{\frac{n-p+1}{n}}\, dm(z) \lesssim \tau(z_k)^{2p} \tag{16}$$

to obtain the desired estimate

$$\|F\|_{A^p(w)}^p \leqslant \sum_k |a_k|^p.$$

It follows from (11) that

$$\int\limits_{|z-z_k|<\tau(z_k)} \left|F_{z_k,n,p}(z)\right|^{\frac{p(n-p+1)}{n}} \tau(z)^{2p-2} w(z)^{\frac{n-p+1}{n}}\, dm(z)$$
$$\asymp \int\limits_{|z-z_k|<\tau(z_k)} \tau(z)^{2p-2}\, dm(z) \asymp \tau(z_k)^{2p}. \tag{17}$$

On the other hand, using (8), it follows that

$$\tau(z) \leqslant C 2^j \tau(z_k) \quad \text{if } |z-z_k| < 2^j \tau(z_k).$$

Thus, since $n \geqslant p$, bearing in mind (12), we deduce that

$$\int\limits_{|z-z_k|\geqslant\tau(z_k)} \left|F_{z_k,n,p}(z)\right|^{\frac{p(n-p+1)}{n}} \tau(z)^{2p-2} w(z)^{\frac{n-p+1}{n}}\, dm(z)$$
$$\lesssim \tau(z_k)^{3p(n-p+1)} \int\limits_{|z-z_k|\geqslant\tau(z_k)} \frac{\tau(z)^{2p-2}}{|z-z_k|^{3p(n-p+1)}}\, dm(z)$$
$$\lesssim \tau(z_k)^{3p(n-p+1)} \sum_{j=0}^{\infty} \int\limits_{2^j\tau(z_k)\leqslant|z-z_k|<2^{j+1}\tau(z_k)} \frac{\tau(z)^{2p-2}}{|z-z_k|^{3p(n-p+1)}}\, dm(z)$$
$$\lesssim \sum_{j=0}^{\infty} 2^{-3jp(n-p+1)} \int\limits_{2^j\tau(z_k)\leqslant|z-z_k|<2^{j+1}\tau(z_k)} \tau(z)^{2p-2}\, dm(z)$$
$$\lesssim \tau(z_k)^{2p} \sum_{j=0}^{\infty} 2^{-jp(3n-3p+1)} \lesssim \tau(z_k)^{2p},$$

which together with (17) gives (16). This finishes the proof. □

4. Proof of Theorem 1

4.1. *Proof of (I): boundedness*

Suppose first that $I_d : A^p(w) \to L^q(\mu)$ is bounded. It is enough to deal with the case $|a| \geqslant \rho_0$, where $\rho_0 \in (0,1)$ is the number given in Lemma 3.1. For $a \in \mathbb{D}$ with $|a| \geqslant \rho_0$, consider the function $F_{a,n_0,p}$ obtained in Lemma 3.1, where n_0 is the smallest natural number such that $n_0 p \geqslant 1$. By Corollary 1, we have $\|F_{a,n_0,p}\|_{A^p(w)}^p \asymp \tau(a)^2$. Then, using the estimate (11) of Lemma 3.1,

$$\begin{aligned}\int\limits_{D(\delta\tau(a))} w(z)^{-q/p}\,d\mu(z) &\lesssim \int\limits_{D(\delta\tau(a))} \left|F_{a,n_0,p}(z)\right|^q d\mu(z)\\ &\leqslant \int\limits_{\mathbb{D}} \left|F_{a,n_0,p}(z)\right|^q d\mu(z)\\ &\lesssim \|I_d\|^q_{A^p(w)\to L^q(\mu)} \|F_{a,n_0,p}\|^q_{A^p(w)}\\ &\lesssim \|I_d\|^q_{A^p(w)\to L^q(\mu)} \tau(a)^{\frac{2q}{p}},\end{aligned}$$

proving that $K_{\mu,w} \leqslant C\|I_d\|^q_{A^p(w)\to L^q(\mu)}$.

Conversely, suppose that (1) holds. This implication was proved by Oleinik in [21], but we give a proof here for the sake of completeness. Using Lemma A, and Lemmas 2.2 and 2.1, it follows that

$$\begin{aligned}\int\limits_{\mathbb{D}} \left|f(z)\right|^q d\mu(z) &\leqslant \sum_j \int\limits_{D(\delta\tau(z_j))} \left|f(z)\right|^q d\mu(z)\\ &\lesssim \sum_j \int\limits_{D(\delta\tau(z_j))} \left(\frac{1}{\tau(z)^2} \int\limits_{D(\delta\tau(z))} \left|f(\zeta)\right|^p w(\zeta)\,dm(\zeta)\right)^{\frac{q}{p}} w(z)^{-\frac{q}{p}}\,d\mu(z)\\ &\lesssim \sum_j \left(\int\limits_{D(3\delta\tau(z_j))} \left|f(\zeta)\right|^p w(\zeta)\,dm(\zeta)\right)^{\frac{q}{p}} \int\limits_{D(\delta\tau(z_j))} \frac{w(z)^{-\frac{q}{p}}\,d\mu(z)}{\tau(z)^{\frac{2q}{p}}}\\ &\lesssim K_{\mu,w} \sum_j \left(\int\limits_{D(3\delta\tau(z_j))} \left|f(\zeta)\right|^p w(\zeta)\,dm(\zeta)\right)^{\frac{q}{p}}.\end{aligned} \tag{18}$$

Now, using Minkowski inequality and the finite multiplicity N of the covering $\{D(3\delta\tau(z_j))\}$, we have

$$\begin{aligned}\int\limits_{\mathbb{D}} \left|f(z)\right|^q d\mu(z) &\lesssim K_{\mu,w}\left(\sum_j \int\limits_{D(3\delta\tau(z_j))} \left|f(\zeta)\right|^p w(\zeta)\,dm(\zeta)\right)^{q/p}\\ &\lesssim K_{\mu,w} N^{q/p} \|f\|^q_{A^p(w)},\end{aligned}$$

proving that $I_d : A^p(w) \to L^q(\mu)$ is continuous with $\|I_d\|^q_{A^p(w)\to L^q(\mu)} \lesssim K_{\mu,w}$.

4.2. Proof of (I): compactness

Suppose that (2) holds. Fixed $\delta \in (0, m_\tau)$, consider the covering $\{D(\delta\tau(z_j))\}$ given by Lemma A, and let $\{f_n\}$ be a bounded sequence in $A^p(w)$. By Lemma 2.2, $\{f_n\}$ is uniformly bounded on compact sets, and by Montel's theorem $\{f_n\}$ is a normal family. Then we may extract a subsequence $\{f_{n_k}\}$ converging uniformly on compact sets of $\mathbb{D}$ to some function f. Using Fatou's lemma, it is easy to see that f must be in $A^p(w)$. Given $\varepsilon > 0$, fix $0 < r_0 < 1$ with

$$\sup_{|a|>r_0} \frac{1}{\tau(a)^{2q/p}} \int\limits_{D(\delta\tau(a))} w(z)^{-q/p}\, d\mu(z) < \varepsilon. \tag{19}$$

Observe that there is $r_0' < 1$ with $r_0 \leqslant r_0'$ such that if a point z_k of the sequence $\{z_j\}$ belongs to $\{|z| \leqslant r_0\}$, then $D(\delta\tau(z_k)) \subset \{|z| \leqslant r_0'\}$. So, take n_k big enough such that $\sup_{|z|\leqslant r_0'} |f_{n_k}(z) - f(z)| < \varepsilon$. Then, setting $g_{n_k} = f_{n_k} - f$, and arguing as in (18), it follows that

$$\begin{aligned}
\|g_{n_k}\|_{L^q(\mu)}^q &\leqslant \int\limits_{|z|\leqslant r_0'} \left|g_{n_k}(z)\right|^q d\mu(z) + \sum_{|z_j|>r_0} \int\limits_{D(\delta\tau(z_j))} \left|g_{n_k}(z)\right|^q d\mu(z) \\
&\leqslant \sup_{|z|\leqslant r_0'} \left|g_{n_k}(z)\right|^q \mu(\mathbb{D}) + \sum_{|z_j|>r_0} \int\limits_{D(\delta\tau(z_j))} \left|g_{n_k}(z)\right|^q d\mu(z) \\
&\leqslant C\varepsilon + C\|g_{n_k}\|_{A^p(w)}^q \sup_{|z_j|>r_0} \frac{1}{\tau(z_j)^{2q/p}} \int\limits_{D(\delta\tau(z_j))} w(z)^{-q/p}\, d\mu(z) \\
&< C\varepsilon.
\end{aligned}$$

In the last inequality we use (19) and the fact that $\{f_{n_k} - f\}$ is also a bounded sequence in $A^p(w)$. This proves that $I_d : A^p(w) \to L^q(\mu)$ is compact.

Conversely, suppose that $I_d : A^p(w) \to L^q(\mu)$ is compact. Take the smallest natural number n_0 such that $n_0 p \geqslant 1$ and let

$$f_{a,n_0,p}(z) = \frac{F_{a,n_0,p}(z)}{\tau(a)^{2/p}}, \quad \rho_0 \leqslant |a| < 1,$$

where $\rho_0 \in (0,1)$ and $F_{a,n_0,p}$ are obtained from Lemma 3.1. By Corollary 1,

$$\sup_{|a|\geqslant\rho_0} \|f_{a,n_0,p}\|_{A^p(w)} \leqslant C,$$

and the compactness of the identity operator implies that $\{f_{a,n_0,p}\colon \rho_0 \leqslant |a| < 1\}$ is a compact set in $L^q(\mu)$. Thus

$$\lim_{r\to 1} \int\limits_{r<|z|<1} \left|f_{a,n_0,p}(z)\right|^q d\mu(z) = 0 \quad \text{uniformly in } a. \tag{20}$$

On the other hand, the estimate (12) gives

$$\left|f_{a,n_0,p}(z)\right|^p w(z) \lesssim \frac{\tau(a)^{3n_0p-2}}{(1-r)^{3n_0p}}, \quad |z| \leqslant r,\ |a| \geqslant \frac{1+r}{2}.$$

Thus $f_{a,n_0,p} \to 0$ as $|a| \to 1$ uniformly on compact subsets of $\mathbb{D}$, which together with (20) implies that $\lim_{|a|\to 1^-} \|f_{a,n_0,p}\|_{L^q(\mu)} = 0$. Therefore, using the estimate (11) of Lemma 3.1,

$$\sup_{|a|>r} \frac{1}{\tau(a)^{2q/p}} \int_{D(\delta\tau(a))} w(z)^{-q/p}\, d\mu(z) \lesssim \sup_{|a|>r} \int_{D(\delta\tau(a))} \left|f_{a,n_0,p}(z)\right|^q d\mu(z)$$
$$\leqslant \sup_{|a|>r} \|f_{a,n_0,p}\|^q_{L^q(\mu)},$$

and this tends to zero as $r \to 1^-$, completing the proof.

4.3. Proof of (II)

The implication (a) $\Rightarrow$ (b) is obvious. To prove that (b) $\Rightarrow$ (c), we use an argument of Luecking (see [20]). Let $\{z_k\}$ be the sequence of points in $\mathbb{D}$ from Lemma A. Let n be a positive integer such that $n \geqslant \max(1/p, p)$, and for an arbitrary sequence $\{a_k\} \in \ell^p$, consider the function

$$G_t(z) = \sum_{z_k:\ |z_k|\geqslant\rho_0} a_k r_k(t) \frac{F_{z_k,n,p}(z)}{\tau(z_k)^{2/p}}, \quad 0 < t < 1,$$

where $r_k(t)$ is a sequence of Rademacher functions (see p. 336 of [20], or Appendix A of [12]). By Proposition 2, the function G_t belongs to $A^p(w)$ with

$$\|G_t\|_{A^p(w)} \leqslant C\left(\sum_k |a_k|^p\right)^{1/p}.$$

Thus, condition (b) gives

$$\int_{\mathbb{D}} \left|G_t(z)\right|^q d\mu(z) \leqslant C\left(\sum_k |a_k|^p\right)^{q/p}, \quad 0 < t < 1.$$

Integrating with respect to t from 0 to 1, applying Fubini's theorem, and invoking Khinchine's inequality (see [20]), we obtain

$$\int_{\mathbb{D}} \left(\sum_{z_k:\ |z_k|\geqslant\rho_0} |a_k|^2 \frac{|F_{z_k,n,p}(z)|^2}{\tau(z_k)^{4/p}}\right)^{q/2} d\mu(z) \leqslant C\left(\sum_k |a_k|^p\right)^{q/p}. \tag{21}$$

Let $\delta \in (0, m_\tau)$. If $\chi_E(z)$ denotes the characteristic function of a set E, bearing in mind the estimate (11), and the finite multiplicity N of the covering $\{D(3\delta\tau(z_k))\}$ (see (iv) of Lemma A), we have

$$\sum_{z_k:\ |z_k|\geqslant\rho_0} \frac{|a_k|^q}{\tau(z_k)^{\frac{2q}{p}}} \int_{D(3\delta\tau(z_k))} w(z)^{-q/p}\, d\mu(z)$$
$$\lesssim \sum_{z_k:\ |z_k|\geqslant\rho_0} \frac{|a_k|^q}{\tau(z_k)^{\frac{2q}{p}}} \int_{D(3\delta\tau(z_k))} \left|F_{z_k,n,p}(z)\right|^q d\mu(z)$$

$$= \int\limits_{\mathbb{D}} \sum_{z_k:\ |z_k|\geqslant \rho_0} \frac{|a_k|^q}{\tau(z_k)^{\frac{2q}{p}}} \left|F_{z_k,n,p}(z)\right|^q \chi_{D(3\delta\tau(z_k))}(z)\, d\mu(z)$$

$$\leqslant \max\left\{1, N^{1-q/2}\right\} \int\limits_{\mathbb{D}} \left(\sum_{z_k:\ |z_k|\geqslant \rho_0} |a_k|^2 \frac{|F_{z_k,n,p}(z)|^2}{\tau(z_k)^{4/p}} \right)^{q/2} d\mu(z).$$

This, together with (21) yields

$$\sum_{|z_k|\geqslant \rho_0} \frac{|a_k|^q}{\tau(z_k)^{\frac{2q}{p}}} \int\limits_{D(3\delta\tau(z_k))} w(z)^{-q/p}\, d\mu(z) \leqslant C\left(\sum_k |a_k|^p\right)^{q/p}.$$

Since the sequence $\{a_k\} \in \ell^p$ is arbitrary and $\frac{p}{q} > 1$, if we put $b_k = |a_k|^q$, then using the duality between $\ell^{\frac{p}{q}}$ and $\ell^{\frac{p}{p-q}}$ we conclude that the sequence

$$\left\{ \frac{1}{\tau(z_k)^{\frac{2q}{p}}} \int\limits_{D(3\delta\tau(z_k))} w(z)^{-q/p}\, d\mu(z) \right\}_{|z_k|\geqslant \rho_0}$$

belongs to $\ell^{\frac{p}{p-q}}$, that is

$$\sum_{|z_k|\geqslant \rho_0} \left(\frac{1}{\tau(z_k)^2} \int\limits_{D(3\alpha\tau(z_k))} w(z)^{-q/p}\, d\mu(z) \right)^{\frac{p}{p-q}} \tau(z_k)^2 < \infty. \tag{22}$$

Note that there is $\rho_1 < 1$, with $\rho_0 \leqslant \rho_1$ such that if a point z_k of the sequence $\{z_j\}$ belongs to $\{|z| < \rho_0\}$, then $D(\delta\tau(z_k)) \subset \{|z| < \rho_1\}$. Therefore, using Lemma 2.1, (ii) and (iii) of Lemma A, and (22) we deduce that

$$\int\limits_{|z|\geqslant \rho_1} \left(\frac{1}{\tau(z)^2} \int\limits_{D(\delta\tau(z))} w(\zeta)^{-q/p}\, d\mu(\zeta) \right)^{\frac{p}{p-q}} dm(z)$$

$$\leqslant \sum_{|z_k|\geqslant \rho_0} \int\limits_{D(\delta\tau(z_k))} \left(\frac{1}{\tau(z)^2} \int\limits_{D(\delta\tau(z))} w(\zeta)^{-q/p}\, d\mu(\zeta) \right)^{\frac{p}{p-q}} dm(z)$$

$$\lesssim \sum_{|z_k|\geqslant \rho_0} \left(\frac{1}{\tau(z_k)^2} \right)^{\frac{p}{p-q}} \int\limits_{D(\delta\tau(z_k))} \left(\int\limits_{D(\delta\tau(z))} w(\zeta)^{-q/p}\, d\mu(\zeta) \right)^{\frac{p}{p-q}} dm(z)$$

$$\lesssim \sum_{|z_k|\geqslant \rho_0} \left(\frac{1}{\tau(z_k)^2} \int\limits_{D(3\delta\tau(z_k))} w(\zeta)^{-q/p}\, d\mu(\zeta) \right)^{\frac{p}{p-q}} \tau(z_k)^2 < \infty.$$

This, together with the fact that the integral

$$\int\limits_{|z|<\rho_1} \left(\frac{1}{\tau(z)^2} \int\limits_{D(\delta\tau(z))} w(\zeta)^{-q/p}\, d\mu(\zeta)\right)^{\frac{p}{p-q}} dm(z)$$

is clearly finite, proves that (c) holds.

Finally, we are going to prove that (c) implies (a). The argument is standard, but it is included for the sake of completeness. It is enough to prove that if $\{f_n\}$ is a bounded sequence in $A^p(w)$ that converges to 0 uniformly on compact subsets of $\mathbb{D}$ then $\lim_{n\to\infty}\|f_n\|_{L^q(d\mu)} = 0$.

Let $\delta \in (0, m_\tau)$. Bearing in mind (7), we deduce that for any $r > \frac{1}{3}$

$$D\left(\frac{\delta}{2}\tau(z)\right) \subset \left\{\zeta \in \mathbb{D}\colon\ |\zeta| > \frac{r}{2}\right\}, \quad \text{if } |z| > r. \tag{23}$$

On the other hand, it follows from Lemma 2.2 that

$$\left|f_n(z)\right|^q \leqslant C\frac{w(z)^{-q/p}}{\tau(z)^2} \int\limits_{D(\frac{\delta}{2}\tau(z))} \left|f_n(\zeta)\right|^q w(\zeta)^{q/p}\, dm(\zeta).$$

Integrate respect to $d\mu$, apply Fubini's theorem, use (23) and Lemma 2.1 to obtain

$$\begin{aligned}
&\int\limits_{\{z\in\mathbb{D}\colon |z|>r\}} \left|f_n(z)\right|^q d\mu(z) \\
&\leqslant C \int\limits_{\{z\in\mathbb{D}\colon |z|>r\}} \left(\frac{1}{\tau(z)^2} \int\limits_{D(\frac{\delta}{2}\tau(z))} \left|f_n(\zeta)\right|^q w(\zeta)^{q/p}\, dm(\zeta)\right) w(z)^{-q/p}\, d\mu(z) \\
&\leqslant C \int\limits_{\{\zeta\in\mathbb{D}\colon |\zeta|>\frac{r}{2}\}} \left|f_n(\zeta)\right|^q w(\zeta)^{q/p} \left(\int\limits_{D(\delta\tau(\zeta))} \frac{w(z)^{-q/p}}{\tau(z)^2}\, d\mu(z)\right) dm(\zeta) \\
&\leqslant C \int\limits_{\{\zeta\in\mathbb{D}\colon |\zeta|>\frac{r}{2}\}} \left|f_n(\zeta)\right|^q w(\zeta)^{q/p} \left(\frac{1}{\tau(\zeta)^2} \int\limits_{D(\delta\tau(\zeta))} w(z)^{-q/p}\, d\mu(z)\right) dm(\zeta).
\end{aligned} \tag{24}$$

If condition (c) holds, then for any fixed $\varepsilon > 0$, there is $r_0 \in (0, 1)$, such that

$$\int\limits_{\{\zeta\in\mathbb{D}\colon |\zeta|>\frac{r_0}{2}\}} \left(\frac{1}{\tau(\zeta)^2} \int\limits_{D(\delta\tau(\zeta))} w(z)^{-q/p}\, d\mu(z)\right)^{\frac{p}{p-q}} dm(\zeta) < \varepsilon^{\frac{p}{p-q}}.$$

Then (24) and an application of Hölder's inequality yields

$$
\begin{aligned}
&\int\limits_{\{\zeta\in\mathbb{D}:\ |z|>r_0\}} \left|f_n(z)\right|^q \, d\mu(z) \\
&\leqslant C\|f_n\|^q_{A^p(w)}\left(\int\limits_{\{\zeta\in\mathbb{D}:\ |\zeta|>\frac{r_0}{2}\}}\left(\frac{1}{\tau(\zeta)^2}\int\limits_{D(\delta\tau(\zeta))} w(z)^{-q/p}\, d\mu(z)\right)^{\frac{p}{p-q}} dm(\zeta)\right)^{\frac{p-q}{p}} \\
&\leqslant C\varepsilon \sup\|f_n\|^q_{A^p(w)} \\
&\leqslant C\varepsilon. \qquad (25)
\end{aligned}
$$

Moreover, since $\{|z|\leqslant r_0\}$ is a compact subset of $\mathbb{D}$, we have

$$
\lim_{n\to\infty}\int\limits_{|z|\leqslant r_0}\left|f_n(z)\right|^q\, d\mu(z)=0,
$$

which together with (25), gives

$$
\lim_{n\to\infty}\|f_n\|_{L^q(d\mu)}=0.
$$

This completes the proof of Theorem 1.

5. Proof of Theorem 2

5.1. *Proof of (I) and (II)*

Let $0<p\leqslant q<\infty$, and let $\delta\in(0,m_\tau)$. Since $T_g f(0)=0$ and $(T_g f)'(z)=f(z)g'(z)$, Theorem B gives

$$
\|T_g f\|^q_{A^q(w)} \asymp \int\limits_{\mathbb{D}}\left|f(z)\right|^q\left|g'(z)\right|^q\psi_w(z)^q w(z)\, dm(z).
$$

Therefore, the boundedness of the integration operator $T_g: A^p(w)\to A^q(w)$ is equivalent to the continuity of the embedding $I_d: A^p(w)\to L^q(\mu_{g,w})$ with

$$
d\mu_{g,w}(z)=\left|g'(z)\right|^q\psi_w(z)^q w(z)\, dm(z). \qquad (26)
$$

By Theorem 1, this holds if and only if

$$
\sup_{a\in\mathbb{D}} I(q,p,a)<\infty, \qquad (27)
$$

where

$$
I(q,p,a)=\frac{1}{\tau(a)^{2q/p}}\int\limits_{D(\delta\tau(a))}\left|g'(z)\right|^q\psi_w(z)^q w(z)^{1-q/p}\, dm(z).
$$

Suppose that $T_g : A^p(w) \to A^q(w)$ is bounded. Note that, bearing in mind that $\tau(z)^2 \asymp (\Delta\varphi(z))^{-1}$ and using Lemma 2.1, condition (6) gives

$$\psi_w(z) \asymp \frac{\tau(z)^2}{(1-|z|)^t} \asymp \frac{\tau(a)^2}{(1-|a|)^t} \asymp \psi_w(a) \quad \text{if } z \in D\big(\delta\tau(a)\big). \tag{28}$$

Write $s = \frac{1}{p} - \frac{1}{q}$. Then, using Lemma 2.2 (with $\beta = 1 - \frac{q}{p}$) and Corollary 1 we obtain

$$\begin{aligned}
\big(\psi_w(a)\|K_a\|_{A^2(w)}^{2s}\big|g'(a)\big|\big)^q &\lesssim \frac{\psi_w(a)^q \|K_a\|_{A^2(w)}^{2qs}}{\tau(a)^2 w(a)^{1-\frac{q}{p}}} \int\limits_{D(\delta\tau(a))} \big|g'(z)\big|^q w(z)^{1-\frac{q}{p}}\, dm(z) \\
&\lesssim \frac{\|K_a\|_{A^2(w)}^{2qs}}{\tau(a)^2 w(a)^{1-\frac{q}{p}}} \int\limits_{D(\delta\tau(a))} w(z)^{-\frac{q}{p}}\, d\mu_{g,w}(z) \\
&\lesssim \frac{1}{\tau(a)^{2q/p}} \int\limits_{D(\delta\tau(a))} w(z)^{-\frac{q}{p}}\, d\mu_{g,w}(z) = I(q,p,a).
\end{aligned}$$

Thus, if $T_g : A^p(w) \to A^q(w)$ is bounded, it follows from (27) that

$$\sup_{a\in\mathbb{D}} \psi_w(a)\|K_a\|_{A^2(w)}^{2s}\big|g'(a)\big| < \infty. \tag{29}$$

If $q = p$ then $s = 0$, and (29) proves that if T_g is bounded on $A^p(w)$, then

$$\sup_{a\in\mathbb{D}} \psi_w(a)\big|g'(a)\big| < \infty. \tag{30}$$

Conversely, if (30) holds, then it follows directly that $\sup_{a\in\mathbb{D}} I(p,p,a) < \infty$. Thus T_g is bounded on $A^p(w)$, and the proof of part (a) of (I) is complete.

If $0 < p < q < \infty$, we are going to show that condition (29) implies $g' \equiv 0$. To prove this, it is enough to see that $\psi_w(a)\|K_a\|_{A^2(w)}^{2s}$ goes to infinity as $|a| \to 1^-$. By Corollary 1 and condition (6)

$$\psi_w(a)\|K_a\|_{A^2(w)}^{2s} \asymp \frac{\tau(a)^{2-2s}}{(1-|a|)^t w(a)^s},$$

and this tends to infinity as $|a| \to 1^-$ by Lemma 2.3. This finishes the proof of (II), since the other implication is trivial.

Concerning the compactness part (b) of (I), note that using Theorem B, the compactness of the operator T_g on $A^p(w)$ is equivalent to the compactness of the embedding $I_d : A^p(w) \to L^p(\mu_{g,w})$, where $\mu_{g,w}$ is the measure defined by (26) with $q = p$. By part (I) of Theorem 1, this holds if and only if

$$\lim_{r\to 1^-} \sup_{|a|>r} \frac{1}{\tau(a)^2} \int\limits_{D(\delta\tau(a))} \big|g'(z)\big|^p \psi_w(z)^p\, dm(z) = 0,$$

and proceeding as in the boundedness part, we see that this is equivalent to

$$\lim_{r\to 1^-}\sup_{|a|>r}\psi_w(a)\left|g'(a)\right|=0.$$

5.2. Proof of (III)

By Theorem B one has

$$\|T_g f\|^q_{A^q(w)}\asymp\int_{\mathbb{D}}\left|f(z)\right|^q\left|g'(z)\right|^q\psi_w(z)^q w(z)\,dm(z). \tag{31}$$

Using (31), we have that $T_g: A^p(w)\to A^q(w)$ is bounded if and only the embedding $I_d: A^p(w)\to L^q(\mu_{g,w})$ is bounded, where $\mu_{g,w}$ is the measure defined by (26). Thus the equivalence (a) $\Leftrightarrow$ (b) follows from part (II) of Theorem 1.

(b) $\Rightarrow$ (c). We also use part (II) of Theorem 1 to assert that (b) is equivalent to

$$C_{g,w}:=\int_{\mathbb{D}}\left(\frac{1}{\tau(z)^2}\int_{D(\tau(z))}\left|g'(\zeta)\right|^q\psi_w(\zeta)^q w(\zeta)^{\frac{p-q}{p}}\,dm(\zeta)\right)^{\frac{p}{p-q}}dm(z)<\infty.$$

Now, using Theorem B, Lemma 2.2 and (28), we obtain

$$\begin{aligned}\|g\|^r_{A^r(w)}&\asymp\left|g(0)\right|^r+\int_{\mathbb{D}}\left(\left|g'(z)\right|^q\psi_w(z)^q w(z)^{q/r}\right)^{r/q}dm(z)\\&\lesssim\left|g(0)\right|^r+C_{g,w}.\end{aligned}$$

(c) $\Rightarrow$ (b). If $g\in A^r(w)$, then (31), Hölder's inequality and Theorem B gives

$$\begin{aligned}\|T_g f\|^q_{A^q(w)}&\lesssim\left(\int_{\mathbb{D}}\left|f(z)\right|^p w(z)\,dm(z)\right)^{q/p}\left(\int_{\mathbb{D}}\left|g'(z)\right|^r\psi_w(z)^r w(z)\,dm(z)\right)^{q/r}\\&\lesssim\|g\|^q_{A^r(w)}\|f\|^q_{A^p(w)}.\end{aligned}$$

Thus $T_g: A^p(w)\to A^q(w)$ is bounded with $\|T_g\|\lesssim\|g\|_{A^r(w)}$. This finishes the proof.

6. Schatten classes on $A^2(w)$

If $\{e_n\}$ is an orthonormal basis of a Hilbert space H of analytic functions in $\mathbb{D}$ with reproducing kernel K_z, then

$$K_z(\zeta)=\sum_n e_n(\zeta)\overline{e_n(z)}$$

for all z and ζ in $\mathbb{D}$ (see e.g. [28, Theorem 4.19]). It follows that

$$\sum_n \left|e_n(z)\right|^2 \leqslant \|K_z\|_H^2 \tag{32}$$

for any orthonormal set $\{e_n\}$ of H. Also, by (6) we have

$$\frac{\partial}{\partial \bar{z}} K_z(\zeta) = \sum_n \overline{e_n'(z)} e_n(\zeta), \quad z, \zeta \in \mathbb{D}.$$

Thus Parseval's identity gives

$$\left\| \frac{\partial}{\partial \bar{z}} K_z \right\|_H^2 = \sum_n \left|e_n'(z)\right|^2. \tag{33}$$

Now, we are going to give the proof of Theorem 3 on the description of the Schatten classes $\mathcal{S}_p := \mathcal{S}_p(A^2(w))$. First we consider the sufficiency part of the case $1 < p < \infty$. We need the following lemma.

Lemma 6.1. *Let $w \in \mathcal{W}$ satisfying* (6). *Then*

$$\left\| \frac{\partial}{\partial \bar{z}} K_z \right\|_{A^2(w)} = \mathrm{O}\left(\frac{\|K_z\|_{A^2(w)}}{\psi_w(r)} \right), \quad |z| = r.$$

Proof. Let $\{e_n\}$ be the orthonormal basis of $A^2(w)$ given by

$$e_n(z) = z^n \delta_n^{-1}, \quad n \in \mathbb{N},$$

where $\delta_n^2 = 2\int_0^1 r^{2n+1} w(r)\,dr$. Using Corollary 1 and (6), we have that

$$\sum_{n=0}^{\infty} r^{2n} \delta_n^{-2} = \sum_{n=0}^{\infty} \left|e_n(z)\right|^2 = \|K_z\|_{A^2(w)}^2 \asymp (1-r)^{-t} \left(\int_r^1 w(s)\,ds \right)^{-1},$$

for some $t \geqslant 1$. So, if we consider the analytic function in $\mathbb{D}$ defined by

$$f(z) = \sum_{n=0}^{\infty} z^n \delta_n^{-1},$$

then $M_2(r, f) = (\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2\, d\theta)^{\frac{1}{2}} \asymp \Phi(r)$, as $r \to 1^-$, where

$$\Phi(r) = (1-r)^{-t/2} \left(\int_r^1 w(s)\,ds \right)^{-1/2}.$$

Now, bearing in mind that $\psi_w(r) \leqslant (1-r)$ for a decreasing weight w, a calculation shows that

$$\Phi'(r) \asymp \frac{\Phi(r)}{\psi_w(r)}, \quad r \to 1^-.$$

Moreover, it is easy to see that for $w \in \mathcal{W}$

$$\limsup_{r \to 1^-} \frac{\Phi''(r)\Phi(r)}{(\Phi'(r))^2} < \infty. \tag{34}$$

Thus we can apply Theorem 2.1 of [23], which says that

$$M_2\big(r, f'\big) = \mathrm{O}\big(\Phi'(r)\big) \quad \text{if } M_2(r, f) = \mathrm{O}\big(\Phi(r)\big),$$

when (34) holds (see condition (3.3) of [23]). Finally, since for $r = |z|$,

$$\left\| \frac{\partial}{\partial \bar{z}} K_z \right\|^2_{A^2(w)} = \sum_{n=0}^{\infty} \left| e_n'(z) \right|^2 = \sum_{n=1}^{\infty} n^2 r^{2n-2} \delta_n^{-2} = M_2^2\big(r, f'\big)$$

we obtain

$$\left\| \frac{\partial}{\partial \bar{z}} K_z \right\|_{A^2(w)} = M_2\big(r, f'\big) = \mathrm{O}\big(\Phi'(r)\big) \asymp \frac{\Phi(r)}{\psi_w(r)} \asymp \frac{\|K_z\|}{\psi_w(z)}, \quad r \to 1^-. \qquad \square$$

Proposition 3. *Let $g \in H(\mathbb{D})$, $1 < p < \infty$ and $w \in \mathcal{W}$ satisfying* (6). *If $\psi_w |g'| \in L^p(\mathbb{D}, \Delta\varphi \, dm)$ then $T_g \in \mathcal{S}_p(A^2(w))$.*

Proof. By Theorem B, the inner product

$$\langle f, g \rangle_* = f(0)\overline{g(0)} + \int_{\mathbb{D}} f'(z)\overline{g'(z)} \psi_w(z)^2 w(z) \, dm(z)$$

gives a norm on $A^2(w)$ equivalent to the usual one. If $1 < p < \infty$, the operator T_g belongs to the Schatten p-class $\mathcal{S}_p$ if and only if

$$\sum_n \left| \langle T_g e_n, e_n \rangle_* \right|^p < \infty$$

for any orthonormal set $\{e_n\}$ (see [28, Theorem 1.27]). Let $\{e_n\}$ be an orthonormal set of $(A^2(w), \langle , \rangle_*)$. Note that Theorem B gives

$$\int_{\mathbb{D}} \left| e_n(z) e_n'(z) \right| \psi_w(z) w(z) \, dm(z) \asymp \left\| e_n^2 \right\|_{A^1(w)} \asymp \|e_n\|^2_{A^2(w)} = 1.$$

This together with Hölder's inequality yields

$$\begin{aligned}\sum_n \bigl|\langle T_g e_n, e_n\rangle_*\bigr|^p &\leqslant \sum_n \biggl(\int_{\mathbb{D}} \bigl|g'(z)e_n(z)e_n'(z)\bigr|\psi_w(z)^2 w(z)\,dm(z)\biggr)^p \\ &\lesssim \sum_n \int_{\mathbb{D}} \bigl|g'(z)\bigr|^p \bigl|e_n(z)e_n'(z)\bigr|\psi_w(z)^{p+1} w(z)\,dm(z) \\ &= \int_{\mathbb{D}} \bigl|g'(z)\bigr|^p \biggl(\sum_n \bigl|e_n(z)e_n'(z)\bigr|\biggr)\psi_w(z)^{p+1} w(z)\,dm(z),\end{aligned}$$

and since $\|K_z\|^2_{A^2(w)} w(z) \asymp \Delta\varphi(z)$ (see Corollary 1), the result will be proved if we are able to show that

$$\sum_n \bigl|e_n(z)e_n'(z)\bigr| \lesssim \frac{\|K_z\|^2_{A^2(w)}}{\psi_w(z)}. \tag{35}$$

To prove (35), we use Cauchy–Schwarz inequality together with (32) and (33) to obtain

$$\begin{aligned}\sum_n \bigl|e_n(z)e_n'(z)\bigr| &\leqslant \biggl(\sum_n \bigl|e_n(z)\bigr|^2\biggr)^{1/2} \biggl(\sum_n \bigl|e_n'(z)\bigr|^2\biggr)^{1/2} \\ &\leqslant \|K_z\|_{A^2(w)} \biggl\|\frac{\partial}{\partial \bar{z}} K_z\biggr\|_{A^2(w)}.\end{aligned}$$

Now, the inequality (35) follows from Lemma 6.1. This completes the proof of the proposition. □

Now we turn to show the necessity for the case $0 < p < \infty$.

Proposition 4. *Let $g \in H(\mathbb{D})$, $0 < p < \infty$ and $w \in \mathcal{W}$ satisfying* (6). *If $T_g \in \mathcal{S}_p(A^2(w))$, then $\psi_w|g'| \in L^p(\mathbb{D}, \Delta\varphi\,dm)$.*

Proof. We split the proof in two cases.

Case $2 \leqslant p < \infty$. Suppose that T_g is in $\mathcal{S}_p$, and let $\{e_k\}$ be an orthonormal set in $A^2(w)$ and $n \geqslant \max\{1/p, p\}$. Let $\{z_k\}$ be the sequence from Lemma 3.1, and consider the operator A taking $e_k(z)$ to $f_{z_k}(z) = F_{z_k,n,p}(z)/\tau(z_k)$. It follows from Proposition 2 that the operator A is bounded on $A^2(w)$. Since $\mathcal{S}_p$ is a two-sided ideal in the space of bounded linear operators on $A^2(w)$, then $T_g A$ belongs to $\mathcal{S}_p$ (see [28, p. 27]). Thus, by [28, Theorem 1.33]

$$\sum_k \bigl\|T_g(f_{z_k})\bigr\|^p_{A^2(w)} = \sum_k \|T_g A e_k\|^p_{A^2(w)} < \infty.$$

This together with Lemma 3.1 and Theorem B gives

$$\begin{aligned}
&\sum_k \frac{1}{\tau(z_k)^p}\biggl(\int\limits_{D(\tau(z_k))} \bigl|g'(z)\bigr|^2 \psi_w(z)^2\, dm(z)\biggr)^{p/2} \\
&\quad\asymp \sum_k \biggl(\int\limits_{D(\tau(z_k))} \bigl|f_{z_k}(z)\bigr|^2 \bigl|g'(z)\bigr|^2 \psi_w(z)^2 w(z)\, dm(z)\biggr)^{p/2} \\
&\quad\lesssim \sum_k \biggl(\int\limits_{\mathbb{D}} \bigl|f_{z_k}(z)\bigr|^2 \bigl|g'(z)\bigr|^2 \psi_w(z)^2 w(z)\, dm(z)\biggr)^{p/2} \\
&\quad\asymp \sum_k \bigl\|T_g(f_{z_k})\bigr\|^p_{A^2(w)} < \infty. \qquad (36)
\end{aligned}$$

On the other hand, if δ is sufficiently small, applying Lemmas 2.2, 2.1 and Lemma A and arguing as in (28), it follows that

$$\begin{aligned}
&\int\limits_{\mathbb{D}} \bigl|g'(z)\bigr|^p \psi_w(z)^p \Delta\varphi(z)\, dm(z) \\
&\quad\lesssim \sum_k \int\limits_{D(\delta\tau(z_k))} \biggl(\frac{1}{\tau(z)^2} \int\limits_{D(\delta\tau(z))} \bigl|g'(\zeta)\bigr|^2 dm(\zeta)\biggr)^{p/2} \psi_w(z)^p \frac{dm(z)}{\tau(z)^2} \\
&\quad\lesssim \sum_k \frac{1}{\tau(z_k)^p} \int\limits_{D(\delta\tau(z_k))} \biggl(\int\limits_{\tilde{D}(\delta\tau(z_k))} \bigl|g'(\zeta)\bigr|^2 \psi_w(\zeta)^2\, dm(\zeta)\biggr)^{p/2} \frac{dm(z)}{\tau(z)^2} \\
&\quad\lesssim \sum_k \frac{1}{\tau(z_k)^p} \biggl(\int\limits_{D(3\delta\tau(z_k))} \bigl|g'(\zeta)\bigr|^2 \psi_w(\zeta)^2\, dm(\zeta)\biggr)^{p/2}.
\end{aligned}$$

This together with (36) concludes the proof.

Case $0 < p < 2$. If $T_g \in \mathcal{S}_p$ then the positive operator $T_g^* T_g$ belongs to $\mathcal{S}_{p/2}$. Without loss of generality we may assume that $g' \neq 0$. Suppose

$$T_g^* T_g f = \sum_n \lambda_n \langle f, e_n\rangle e_n$$

is the canonical decomposition of $T_g^* T_g$. Then not only is $\{e_n\}$ an orthonormal set, it is also an orthonormal basis. Indeed, if there is a unit vector $e \in A^2(w)$ such that $e \perp e_n$ for all $n \geqslant 1$, then

$$\int\limits_{\mathbb{D}} \bigl|g'(z)\bigr|^2 \bigl|e(z)\bigr|^2 \psi_w(z)^2 w(z)\, dm(z) \asymp \|T_g e\|^2_{A^2(w)} = \bigl\langle T_g^* T_g e, e\bigr\rangle = 0$$

because $T_g^* T_g$ is a linear combination of the vectors e_n. This would give $g' \equiv 0$.

Since $\{e_n\}$ is an orthonormal basis of $A^2(w)$, then

$$\|K_z\|^2 = \sum_n \left|e_n(z)\right|^2.$$

This identity together with Corollary 1 and Hölder's inequality yields

$$\begin{aligned}
\int_{\mathbb{D}} \left|g'(z)\right|^p \psi_w(z)^p \Delta\varphi(z)\, dm(z) &\asymp \int_{\mathbb{D}} \left|g'(z)\right|^p \psi_w(z)^p \|K_z\|^2 w(z)\, dm(z) \\
&= \sum_n \int_{\mathbb{D}} \left|g'(z)\right|^p \psi_w(z)^p \left|e_n(z)\right|^2 w(z)\, dm(z) \\
&\leqslant \sum_n \left(\int_{\mathbb{D}} \left|g'(z)\right|^2 \psi_w(z)^2 \left|e_n(z)\right|^2 w(z)\, dm(z) \right)^{p/2} \\
&\lesssim \sum_n \langle T_g^* T_g e_n, e_n \rangle^{p/2} = \sum_n \lambda_n^{p/2} = \left\| T_g^* T_g \right\|_{\mathcal{S}_{p/2}}^{p/2}.
\end{aligned}$$

This completes the proof. □

Finally, we shall prove the main result in this section.

Proof of Theorem 3. Part (a) follows directly from Propositions 3 and 4. Moreover, if $0 < p \leqslant 1$ and $T_g \in \mathcal{S}_p(A^2(w))$, Proposition 4, (7) and (6) imply that

$$\begin{aligned}
\int_{\mathbb{D}} \frac{|g'(z)|^p\, dm(z)}{(1-|z|)^{tp}(1-|z|)^{2(1-p)}} &\lesssim \int_{\mathbb{D}} \frac{|g'(z)|^p\, dm(z)}{(1-|z|)^{tp}\tau(z)^{2(1-p)}} \\
&\asymp \int_{\mathbb{D}} \left|g'(z)\right|^p \frac{\Delta\varphi(z)^{1-p}}{(1-|z|)^{tp}}\, dm(z) \\
&\asymp \int_{\mathbb{D}} \left|g'(z)\right|^p \psi_\omega(z)^p \Delta\varphi(z)\, dm(z) < \infty.
\end{aligned}$$

Therefore, it follows that $(t-2)p + 2 \geqslant 1$, and consequently $g' \equiv 0$, which gives (b). The proof is complete. □

7. Some examples of weights in the class $\mathcal{W}$

In this section, several examples of weights in the class $\mathcal{W}$ are given. We check that they satisfy the condition (6) in Theorem 2, and by computing the distorsion functions, we also offer the corresponding description for the boundedness and compactness of the integration operator T_g in each case.

Example 1. The weights

$$w_{\gamma,\alpha}(r)=(1-r)^{\gamma}\exp\left(\frac{-c}{(1-r)^{\alpha}}\right),\quad \gamma\geqslant 0,\ \alpha>0,\ c>0,$$

are in the class $\mathcal{W}$ with associated subharmonic function

$$\varphi_{\gamma,\alpha}(z)=-\gamma\log\bigl(1-|z|\bigr)+c\bigl(1-|z|\bigr)^{-\alpha}.$$

We have that

$$\bigl(\Delta\varphi_{\gamma,\alpha}(z)\bigr)^{-1}\asymp\tau(z)^2=\bigl(1-|z|\bigr)^{2+\alpha},$$

and it is easy to see that $\tau(z)$ satisfies the conditions in the definition of the class $\mathcal{W}$. Also, since $\psi_{w_{\gamma,\alpha}}(r)\asymp(1-r)^{1+\alpha}$ (see e.g. [26, Example 3.2]), (6) is satisfied with $t=1$. In particular, the case $q=p$ of Theorem 2 says that T_g is bounded on $A^p(w_{\gamma,\alpha})$ if and only if

$$\sup_{z\in\mathbb{D}}\bigl(1-|z|\bigr)^{1+\alpha}\bigl|g'(z)\bigr|<\infty,$$

and T_g is compact on $A^p(w_{\gamma,\alpha})$ whenever

$$\lim_{|z|\to 1^-}\bigl(1-|z|\bigr)^{1+\alpha}\bigl|g'(z)\bigr|=0.$$

As mentioned above, this answers a question raised in [4, p. 353]. We also note that, by Theorem 3, the operator T_g belongs to the Schatten p-class of $A^2(w_{\gamma,\alpha})$ if and only if g is in the Dirichlet-type space $\mathcal{D}^p_\beta$ with $\beta=p-2+\alpha(p-1)$, that is, the space of those $g\in H(\mathbb{D})$ with

$$\int_{\mathbb{D}}\bigl|g'(z)\bigr|^p\bigl(1-|z|^2\bigr)^{\beta}\,dm(z)<\infty.$$

Example 2. For $\alpha>1$ and $A>0$ the weights

$$w(r)=\exp\left(-A\left(\log\frac{e}{1-r}\right)^{\alpha}\right),$$

with associated subharmonic function $\varphi(z)=A(\log\frac{e}{1-|z|})^{\alpha}$, belong to the class $\mathcal{W}$. Indeed, it is easy to see that

$$\Delta\varphi(z)\asymp\bigl(1-|z|\bigr)^{-2}\left(\log\frac{e}{1-|z|}\right)^{\alpha-1},\tag{37}$$

so $\tau(z)=(1-|z|)(\log\frac{e}{1-|z|})^{\frac{1-\alpha}{2}}$, and since $\alpha>1$

$$\tau'(r)\asymp\left(\log\frac{e}{1-r}\right)^{\frac{-\alpha+1}{2}},\quad r\to 1^-,$$

which implies that $\lim_{r\to 1^-}\tau(r)=\lim_{r\to 1^-}\tau'(r)=0$. Moreover, the function $\tau(r)(1-r)^{-2}$ increases for r close to 1. This proves that $w\in\mathcal{W}$.

On the other hand, since w has distorsion function (see [26, Example 3.4])

$$\psi_w(r)\asymp\frac{1-r}{(\log\frac{e}{1-r})^{\alpha-1}},$$

then (37) gives $(\Delta\varphi(z))^{-1}\asymp(1-|z|)\psi_w(z)$. Therefore, (6) is satisfied with $t=1$. For this weight, Theorem 2 says that T_g is bounded on $A^p(w)$ if and only if

$$\sup_{z\in\mathbb{D}}\big(1-|z|\big)\left(\log\frac{e}{1-|z|}\right)^{1-\alpha}\big|g'(z)\big|<\infty,$$

and T_g is compact on $A^p(w)$ whenever the corresponding "little oh" condition holds.

Example 3. For $\alpha,\beta,\gamma>0$, the double exponential weight

$$w(r)=\exp\left(-\gamma\exp\left(\frac{\beta}{(1-r)^\alpha}\right)\right)$$

belongs to $\mathcal{W}$. Indeed, the associated subharmonic function is $\varphi(z)=\gamma\exp(\frac{\beta}{(1-|z|)^\alpha})$, and a straightforward computation gives

$$\Delta\varphi(z)\asymp\big(1-|z|\big)^{-2\alpha-2}\exp\left(\frac{\beta}{(1-|z|)^\alpha}\right). \tag{38}$$

Then we can take $\tau(z)=(1-|z|)^{\alpha+1}\exp(\frac{-\beta/2}{(1-|z|)^\alpha})$. Since

$$\tau'(r)\asymp\exp\left(\frac{-\beta/2}{(1-r)^\alpha}\right),\quad r\to 1^-,$$

we obtain $\lim_{r\to 1^-}\tau(r)=\lim_{r\to 1^-}\tau'(r)=0$. Also, it is easy to see that $\lim_{r\to 1^-}\tau'(r)\log\frac{1}{\tau(r)}=0$. This proves that $w\in\mathcal{W}$.

Moreover, since w has distorsion function (see [26, Example 3.3])

$$\psi_w(r)\asymp(1-r)^{1+\alpha}\exp\left(-\frac{\beta}{(1-r)^\alpha}\right),$$

then (38) gives $(\Delta\varphi(z))^{-1}\asymp(1-|z|)^{1+\alpha}\psi_w(z)$. So (6) is satisfied with $t=1+\alpha$. In this example, the case $q=p$ of Theorem 2 says that T_g is bounded on $A^p(w)$ if and only if

$$\sup_{z\in\mathbb{D}}\big(1-|z|\big)^{1+\alpha}\exp\left(-\frac{\beta}{(1-|z|)^\alpha}\right)\big|g'(z)\big|<\infty,$$

and T_g is compact on $A^p(w)$ if the "little oh" condition holds.

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